Transitions between turbulent and laminar superfluid vorticity states in the outer core of a neutron star

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ABSTRACT

We investigate the global transition from a turbulent state of superfluid vorticity (quasi-isotropic vortex tangle) to a laminar state (rectilinear vortex array), and vice versa, in the outer core of a neutron star. By solving numerically the hydrodynamic Hall-Vinen-Bekarevich-Khalatnikov equations for a rotating superfluid in a differentially rotating spherical shell, we find that the meridional counterflow driven by Ekman pumping exceeds the Donnelly-Glaberson threshold throughout most of the outer core, exciting unstable Kelvin waves which disrupt the rectilinear vortex array, creating a vortex tangle. In the turbulent state, the torque exerted on the crust oscillates, and the crust-core coupling is weaker than in the laminar state. This leads to a new scenario for the rotational glitches observed in radio pulsars: a vortex tangle is sustained in the differentially rotating outer core by the meridional counterflow, a sudden spin-up event (triggered by an unknown process) brings the crust and core into corotation, the vortex tangle relaxes back to a rectilinear vortex array (in $\lesssim 10^5$ s), then the crust spins down electromagnetically until enough meridional counterflow builds up (after $\lesssim 1$ yr) to reform a vortex tangle. The turbulent-laminar transition can occur uniformly or in patches; the associated time-scales are estimated from vortex filament theory. We calculate numerically the global structure of the flow with and without an inviscid superfluid component, for Hall-Vinen (laminar) and Gorter-Mellink (turbulent) forms of the mutual friction. We also calculate the post-glitch evolution of the angular velocity of the crust and its time derivative, and compare the results with radio pulse timing data, predicting a correlation between glitch activity and Reynolds number. Terrestrial laboratory experiments are proposed to test some of these ideas.

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1. Introduction

Timing irregularities in a rotation-powered pulsar, such as discontinuous glitches (Lyne et al. 2000; Zou et al. 2004; Shabanova 2005) and stochastic timing noise (Hobbs 2002; Scott et al. 2003), provide an indirect probe of the internal structure of the star. The physical processes usually invoked to explain these phenomena are (un)pinning of Feynman-Onsager vortices in the crystalline inner crust (Anderson & Itoh 1975), starquakes (Ruderman 1976), and thermally driven vortex creep (Alpar et al. 1984b; Link et al. 1993). Less attention has been directed at the *global hydrodynamics* of the superfluid, except within the context of the spin-up problem in cylindrical geometry (Anderson et al. 1978; Reisenegger 1993; Carter et al. 2000). The importance of the global hydrodynamics was demonstrated by Tsakadze & Tsakadze (1980), who simulated pulsar rotational irregularities in the laboratory by impulsively accelerating rotating containers of He II, obtaining qualitative agreement with radio timing data (e.g. glitch amplitudes and post-glitch relaxation times).

In this paper, we examine how the global flow pattern of superfluid in the outer core of a neutron star affects the rotation of the star. We focus on the outer core for simplicity: vortex pinning is thought to be weak or non-existent there (Alpar et al. 1984a; Donati & Pizzochero 2003, 2004), and the fluid is mainly isotropic ($^1S_0$ Cooper pairing) (Sedrakian & Sedrakian 1995; Yakovlev et al. 1999), reducing the problem to a hydrodynamic one in a spherical shell. Even with this simplification, the calculation remains numerically challenging: the spherical Couette problem for a superfluid was solved for the first time only recently (Peralta et al. 2005), generalizing previous work on the cylindrical Taylor-Couette problem for a superfluid (Henderson et al. 1995; Henderson & Barenghi 2004) and the spherical Couette problem for a classical viscous fluid (Marcus & Tuckerman 1987a; Dumas & Leonard 1994).

An isotropic ($^1S_0$-paired) neutron superfluid is described by the two-fluid Hall-Vinen-Bekarevich-Khalatnikov (HVBK) model. In this model, the viscous normal fluid and inviscid superfluid components feel a mutual friction force whose magnitude and direction depends on the distribution of Feynman-Onsager vortices (Hall & Vinen 1956a; Bekarevich & Khalatnikov 1961). The vortices are organized in a rectilinear array if the flow is strictly toroidal, but they evolve into a tangle of reconnecting loops when the counterflow along the rotation axis exceeds a threshold, exciting the Donnelly-Glaberson instability (DGI) (Glaberson et al. 1974; Swanson et al. 1983; Tsubota et al. 2003). Peralta et al. (2005) showed that the DGI is excited in a neutron star under a wide range of conditions, driven by the meridional component of the spherical Couette flow (SCF) in the interior. The mutual friction force changes dramatically during transitions between a vortex array and a vortex tangle (Gorter & Mellink 1949; Vinen 1957; Swanson et al. 1983; Schwarz 1985), affecting the rotational evolution of the star.

In this paper, we propose a phenomenological model for timing irregularities in radio pulsars based on the creation and destruction of a vortex tangle — *superfluid turbulence* (Barenghi et al. 1995; Vinen 2003) — in the outer core of a rotating neutron star. In our scenario, a glitch comprises the following sequence of events. (i) Differential rotation between the outer core and crust of the star, built up over time through electromagnetic spin down, generates a meridional Ekman counterflow in the outer core. We show that the axial counterflow exceeds the DGI threshold, creating a vortex tangle throughout the outer core. The mutual friction in this turbulent state takes the isotropic Gorter-Mellink (GM) form and is much weaker than the mutual friction associated with a rectilinear vortex array. (ii) When the glitch occurs, triggered by an unknown mechanism, the outer core and inner crust suddenly come into corotation and the vortex tangle decays, ultimately converting into a rectilinear vortex array. The decay process lasts $\lesssim 10^5$ s, depending on the drag force acting on the vortex rings, after which the mutual friction takes the anisotropic Hall-Vinen (HV) form (Hall & Vinen 1956a,b) and increases by ~ 5 orders of magnitude, precipitating a "torque crisis". (iii) After the glitch, differential rotation builds up again between the outer core and the crust due to electromagnetic spin down. When the axial counterflow exceeds the DGI threshold, after $\lesssim 1$ yr, the vortex array breaks up again into a tangle and the mutual friction drops sharply. Similar transitions from turbulent to laminar flow in a superfluid have been observed in laboratory experiments where He II, cooled to a few mK, flows around an oscillating microsphere (Niemetz et al. 2002; Schoepe 2004).

The paper is organized as follows. HVBK theory is briefly reviewed in §2, together with the pseudospectral numerical method which we use to solve the HVBK equations in a differentially rotating spherical shell. The physics of the turbulent-laminar transition in a generic glitch scenario is elaborated in §3. The response of the stellar crust to a turbulent-laminar transition in the outer core is calculated numerically in §4. Finally, the results are summarized and applied to observational data in §5; a fuller observational analysis will be carried out in a future paper.

2. HVBK model of the outer core

The neutron superfluid in the outer core exists in an isotropic ($^1S_0$) phase for densities ρ in the range $0.6 < \rho/\rho_* < 1.0$ (where ρ_* is the nuclear saturation density) and an anisotropic

($^3P_2$) phase for $1.0 < \rho/\rho_* < 1.6$ (Yakovlev et al. 1999); the depth at which the $^1S_0$-$^3P_2$ transition occurs is not known precisely (Epstein 1988). As the hydrodynamic equations for the $^3P_2$ phase (Mastrano & Melatos 2005) are complicated and hard to handle numerically, we construct our model around the $^1S_0$ phase, described by HVBK theory in this paper. We plan to extend the model to the $^3P_2$ phase in the future.

The protons in the outer core are probably in a type II superconducting state, where the magnetic field is quantized into fluxoids (Sauls 1989). Protons inside magnetic fluxoids interact with neutrons in Feynman-Onsager vortices (Sauls 1989; Ruderman 1991; Link 2003). If vortex pinning occurs, and if the core magnetic field has comparable poloidal and toroidal components (Thompson & Duncan 1993), the geometry of the neutron-fluxoid interaction can be complicated (Ruderman et al. 1998). To avoid these difficulties, we incorporate the protons (and other charged species) into the normal component of the superfluid, a common approximation (Comer & Joynt 2003; Prix et al. 2004). We neglect proton-neutron entrainment (Mendell 1991; Sedrakian & Sedrakian 1995), which is observed in terrestrial ^{3}He-^{4}He mixtures (Andreev & Bashkin 1976; Andersson & Comer 2001) and is an important mechanism modifying the mutual friction for rectilinear vortices in a neutron star (Mendell 1991; Andersson et al. 2006). We neglect hydrodynamic forces arising from entropy gradients and angular momentum textures (Mastrano & Melatos 2005). Finally, we neglect vortex pinning for simplicity, except at the inner and outer boundaries, even though it seems likely that pinning plays an important role in glitch dynamics (Epstein & Baym 1988; Baym et al. 1992; Hirasawa & Shibazaki 2001; Link & Cutler 2002). [1]

2.1. HVBK theory

The HVBK model for a rotating superfluid is a generalization of the Landau-Tisza two-fluid model that includes the hydrodynamic forces exerted by quantized vortices in He II (Hall & Vinen 1956a; Bekarevich & Khalatnikov 1961) and $^1S_0$-paired neutron matter (Tilley & Tilley 1986; Prix et al. 2004). Fluid particles in the theory are assumed to be threaded by many coaligned vortices, a valid assumption over length-scales longer than the average inter-vortex separation (and shorter than the average radius of curvature). In the continuum limit, the vorticity of the superfluid component satisfies $\boldsymbol{\omega}_s = \nabla \times \mathbf{v}_s \neq 0$ macroscopically (cf. $\nabla \times \mathbf{v}_s = 0$ microscopically). The normal component comprises charged species (protons and electrons, locked together by the external magnetic field) plus thermal excitations and

[1] Three-fluid models analyzing post-glitch relaxation favor pinning at the core boundary (Sedrakian & Sedrakian 1995), e.g. due to vortex cluster-Meissner current interactions (Sedrakian & Cordes 1999).

behaves like a classical, viscous, Navier-Stokes fluid (kinematic viscosity ν_n). The isothermal, incompressible ($\nabla \cdot \mathbf{v}_n = \nabla \cdot \mathbf{v}_s = 0$) HVBK equations of motion take the form (Barenghi & Jones 1988; Henderson & Barenghi 2000)

$$\frac{d_n \mathbf{v}_n}{dt} = -\frac{\nabla p_n}{\rho} + \nu_n \nabla^2 \mathbf{v}_n + \frac{\rho_s}{\rho}\mathbf{F} - \frac{\nu_s \rho_s}{\rho}\nabla|\boldsymbol{\omega}_s|, \tag{1}$$

$$\frac{d_s \mathbf{v}_s}{dt} = -\frac{\nabla p_s}{\rho} + \nu_s \mathbf{T} - \frac{\rho_n}{\rho}\mathbf{F} - \frac{\nu_s \rho_s}{\rho}\nabla|\boldsymbol{\omega}_s|, \tag{2}$$

with $d_{n,s}/dt = \partial/\partial t + \mathbf{v}_{n,s} \cdot \nabla$, where $\mathbf{v}_n$ ($\mathbf{v}_s$) and ρ_n (ρ_s) are the normal fluid (superfluid) velocities and densities respectively. Effective pressures p_s and p_n are defined by $\nabla p_s = \nabla p - \frac{1}{2}\rho_n \nabla(\mathbf{v}_{ns}^2)$ and $\nabla p_n = \nabla p + \frac{1}{2}\rho_s \nabla(\mathbf{v}_{ns}^2)$, with $\mathbf{v}_{ns} = \mathbf{v}_n - \mathbf{v}_s$. Note that, in neutron star applications, the gravitational potential can be subsumed into p without loss of generality, in the incompressible limit.

The vortex tension force per unit mass, which arises from local circulation around quantized vortices (Andronikashvili & Mamaladze 1966), is defined as

$$\nu_s \mathbf{T} = \boldsymbol{\omega}_s \times (\nabla \times \hat{\boldsymbol{\omega}}_s), \tag{3}$$

with $\hat{\boldsymbol{\omega}}_s = \boldsymbol{\omega}_s/|\boldsymbol{\omega}_s|$. Here, $\nu_s = (\kappa/4\pi)\ln(b_0/a_0)$ is the stiffness parameter, $\kappa = h/2m_n$ is the quantum of circulation, m_n is the mass of the neutron, a_0 is the radius of the vortex core, and b_0 is the intervortex spacing. Note that ν_s, which has the dimensions of a kinematic viscosity, controls the oscillation frequency of Kelvin waves excited on vortex lines (Henderson et al. 1995).

The mutual friction force per unit mass, $\mathbf{F}$, arises from the interaction between the quantized vortex lines and the normal fluid (via roton scattering in He II and electron scattering in a neutron star) (Hall & Vinen 1956a,b). The structure of this force depends on the global configuration of the vortices. If the vortices form a rectilinear array, the friction takes the anisotropic HV form (Hall & Vinen 1956a,b)

$$\mathbf{F} = \frac{1}{2}B\hat{\boldsymbol{\omega}}_s \times (\boldsymbol{\omega}_s \times \mathbf{v}_{ns} - \nu_s \mathbf{T}) + \frac{1}{2}B'(\boldsymbol{\omega}_s \times \mathbf{v}_{ns} - \nu_s \mathbf{T}) \tag{4}$$

where B and B' are temperature-dependent mutual friction coefficients (Barenghi et al. 1983). If the vortices form a vortex tangle, the friction takes the isotropic GM form (Gorter & Mellink 1949)

$$\mathbf{F} = A'\left(\frac{\rho_n \rho_s v_{ns}^2}{\kappa \rho^2}\right)\mathbf{v}_{ns}, \tag{5}$$

where $A' = B^3 \rho_n^2 \pi^2 \chi_1^2 / 3\rho^2 \chi_2^2$ is a dimensionless temperature-dependent coefficient, related to the original GM constant (usually denoted by A in the literature) by $A' = A\rho\kappa$. Here, χ_1 and χ_2 are dimensionless constants of order unity (Vinen 1957). The HV and GM forces are in the ratio $\sim 10^5$ under the physical conditions prevailing in the outer core of a neutron star (Peralta et al. 2005). This implies that the normal and superfluid components of the star (and hence the crust and core, through viscous torques) are effectively uncoupled most of the time (when enough differential rotation has built up), but become tightly coupled in the immediate aftermath of a glitch when a turbulent-to-laminar transition occurs (Peralta et al. 2005). It is important to bear this in mind when reading §3.

2.2. Spherical Couette flow

We model the global hydrodynamics of the outer core by considering an HVBK superfluid enclosed in a differentially rotating spherical shell, i.e. superfluid spherical Couette flow. The inner (radius R_1) and outer (radius R_2) surfaces of the shell rotate at angular velocities Ω_1 and Ω_2, respectively, about a common axis. All quantities are expressed in dimensionless form using R_2 as the unit of length and Ω_1^{-1} as the unit of time. We adopt spherical polar coordinates (r, θ, ϕ), with the z direction along the axis of rotation.

The boundary conditions satisfied by the normal and superfluid components are subtle. Neither component can penetrate the boundaries, implying $(\mathbf{v}_n)_r = (\mathbf{v}_s)_r = 0$. The tangential normal fluid velocity satisfies the no-slip Dirichlet condition, viz. $(\mathbf{v}_n)_\theta = 0$, $(\mathbf{v}_n)_\phi = R_{1,2}\Omega_{1,2}\sin\theta$. The behavior of the superfluid at the boundaries is influenced by the quantized vortices, which can either slide past, or pin to, irregularities on the boundaries. The former scenario implies $(\boldsymbol{\omega}_s)_\theta = (\boldsymbol{\omega}_s)_\phi = 0$, while the latter implies $(\boldsymbol{\omega}_s)_r \boldsymbol{\omega}_s \times \mathbf{v}_L = 0$ and requires a knowledge of the vortex line velocity $\mathbf{v}_L$ locally. As the HVBK model provides no information about $\mathbf{v}_L$, we adopt the widely used no-slip compromise $(\mathbf{v}_s)_\theta = 0$, $(\mathbf{v}_s)_\phi = R_{1,2}\Omega_{1,2}\sin\theta$, which in the spherical case does not restrict the orientation of the vortex lines at the surface (Henderson et al. 1995; Henderson & Barenghi 1995, 2004). Note that, initially, $\mathbf{v}_n$ and $\mathbf{v}_s$ must be divergence-free, with $\nabla \times \mathbf{v}_s \neq 0$; the Stokes solution with $\mathbf{v}_s = \mathbf{v}_n$ satisfies this constraint (Landau & Lifshitz 1959). However, the numerical results presented in §3 and §4 are obtained after several Ekman times (and hence rotation periods) have elapsed, by which time the flow has forgotten its initial conditions, as in most astrophysical applications.

Many existing glitch models are based on cylindrical geometries (Anderson et al. 1978; Alpar et al. 1984b; Abney & Epstein 1996; Larson & Link 2002), except for a few experiments with spheres (Tsakadze & Tsakadze 1973, 1980). However, the global flow pattern in spherical Couette flow differs from its cylindrical counterpart in two important ways. First, in a cylinder, the principal flow is toroidal, whereas, in a sphere, the principal flow is an axisymmetric combination of toroidal flow and meridional circulation for all Re (Tuckerman 1983). [2] Second, it is misleading to approximate the equatorial region of a sphere by a cylinder, even for $R_1 \approx R_2$; the curvature, although slight, causes significant differences in the critical Taylor number at which vortices appear (Soward & Jones 1983; Stuart 1986). The differences between cylindrical and spherical Couette flow are equally prominent in classical fluids and superfluids (Peralta et al. 2005).

Spherical Couette flow of a viscous fluid is controlled by three parameters: the Reynolds number $Re = \Omega_1 R_2^2/\nu_n$, the dimensionless gap width $\delta = (R_2 - R_1)/R_2$, and the angular shear $\Delta\Omega = \Omega_2 - \Omega_1$. If a superfluid is also introduced, ν_s emerges as an additional parameter. In the laminar regime ($Re \lesssim 10^4$), the global flow can be classified according to the number of cells of meridional circulation in each hemisphere. In a narrow gap ($\delta < 0.11$), the meridional circulation is slow and can be approximated by $(\mathbf{v}_n)_\theta/(R_2\Delta\Omega) \approx 10^{-2}\delta^2(R_2/R_1)Re$ (Yavorskaya et al. 1986). As Re increases, a centrifugal instability generates toroidal Taylor vortices near the equator until, at the onset of turbulence ($Re \gtrsim 10^4$), a helical traveling wave develops. In a wide gap ($\delta > 0.4$), by contrast, all secondary flows are nonaxisymmetric, and they oscillate when Re exceeds a threshold (Yavorskaya et al. 1986). In classical fluids, these flows have been thoroughly investigated numerically and experimentally (Yavorskaya et al. 1975, 1977; Belayev et al. 1978; Yavorskaya et al. 1986; Marcus & Tuckerman 1987a,b; Junk & Egbers 2000).

Recently, high-resolution numerical simulations of superfluid spherical Couette flow were performed successfully for the first time (Peralta et al. 2005). [3] Figures 1a and 1b display streamlines of the normal fluid and superfluid components for $Re = 3 \times 10^4$, $\delta = 0.3$, $\Delta\Omega = 0.1$, and HV mutual friction. Both components exhibit a similar structure in each hemisphere: a "square" Ekman cell near the pole, three to four secondary meridional cells, a number of smaller cells, and two equatorial vortices near the outer boundary which are created and destroyed intermittently. We emphasize that the normal component is not assumed to be in uniform (solid body) rotation; it is affected by the superfluid (through mutual friction) and the boundary conditions (differential rotation). Figures 1c and 1d display streamlines for a run with identical parameters and GM mutual friction, for which

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[2]Spherical Couette flow is a combination of quasi-parallel-plate Ekman pumping at the pole and quasi-cylindrical, centrifugally driven Taylor vortices at the equator (Junk & Egbers 2000).

[3]Visualizing superfluid spherical Couette flow experimentally is challenging. It has been attempted successfully using glass microspheres in turbulent He II (Bielert & Stamm 1993). The technique of particle image velocimetry is currently being evaluated (Zhang et al. 2004).

the inter-fluid coupling is $\sim 10^5$ times smaller, as noted in §2.1. We observe that the normal fluid exhibits a similar number of vortical structures, which are better developed and more regular than in Figure 1a, and the superfluid flow pattern is nearly unchanged from Figure 1b. Meridional circulation is apparent in all the results; we find axial velocity components in the range $10^{-5} \lesssim (\mathbf{v}_n)_z, (\mathbf{v}_s)_z \lesssim 10^{-2}$.

The flow pattern in Figure 1 changes with temperature just like in cylindrical Couette flow. The superfluid behaves like a classical, viscous fluid near the critical temperature. The Taylor vortices elongate axially, with a more complex pattern of eddies and counter-eddies emerging, as the temperature is lowered (Henderson et al. 1995; Henderson & Barenghi 1995). For $Re \lesssim 268$, anomalous modes emerge in the normal component, as in classical axisymmetric flow (Lorenzen & Mullin 1985), characterized by Ekman cells rotating in the opposite sense to the classical flow (Henderson & Barenghi 2000) except near the critical temperature. For $Re \gtrsim 268$, the tension becomes less important than the mutual friction and the superfluid component comes to resemble the normal component (Henderson & Barenghi 1995).

3. Glitch-induced turbulent-laminar transition

In this section, we investigate how the vorticity state of the outer core changes before, during, and after a glitch, in the context of a generic glitch scenario where the trigger mechanism of the glitch is unspecified.

3.1. Before a glitch: vortex tangle

Laboratory experiments on the attenuation of second sound in narrow channels (Vinen 1957; Swanson et al. 1983), and numerical simulations based on the vortex filament method (Tsubota et al. 2003), show that, in a rotating container of He II, an axial (along z) counterflow $v_{ns} > v_{\rm DG} = 2(2\Omega_2\nu_s)^{1/2}$ excites growing Kelvin waves which destabilize a rectilinear vortex array (Glaberson et al. 1974). The Kelvin waves grow exponentially until reconnection between adjacent vortices occurs, generating a dense tangle.[4] For typical neutron star

[4]The Kelvin waves excited by the DGI are unrelated to the Kelvin waves generated by oscillations of the nuclear lattice in the inner crust (Epstein & Baym 1992).

parameters, one finds

$$v_{\rm DG} = 1.56 \left(\frac{\Omega_*}{10^2\,{\rm rad\,s^{-1}}} \right)^{1/2} \,{\rm cm\,s^{-1}}, \tag{6}$$

taking $\ln(b_0/a_0) = 20$.

The axial component of the Ekman counterflow in the outer core of a typical neutron star generally exceeds the instability threshold (6). This is illustrated by Figure 2, a greyscale plot of $(\mathbf{v}_{ns})_z/v_{\rm DG}$; the DGI is active in regions with $(\mathbf{v}_{ns})_z/v_{\rm DG} \geq 1$. Figure 2a corresponds to HV mutual friction (initially rectilinear vortex array); Figure 2b corresponds to GM mutual friction (after a tangle forms). In both figures, the DGI is active in most of the computational domain, implying that *an initially rectilinear array is disrupted and, once disrupted, stays that way.* This result is extracted empirically from the simulations. The inclusion of compressibility and hence stratification (Abney & Epstein 1996) restricts the DGI region to a thin boundary layer, but the overall conclusion is the same (see §4.5).

The axial counterflow that excites the DGI is driven by Ekman pumping. As the crust spins down electromagnetically at a rate $\dot{\Omega}_*$, differential rotation builds up between the crust and outer core, with

$$\Delta\Omega_{\rm em} = 3.16 \times 10^{-6} \left(\frac{\dot{\Omega}_*}{10^{-13}\,{\rm rad\,s^{-2}}} \right) \left(\frac{t}{1\,{\rm yr}} \right) \,{\rm rad\,s^{-1}}, \tag{7}$$

where t is the time elapsed since the last glitch (Lyne & Graham-Smith 2006; Lyne et al. 2000). The differential rotation induces meridional circulation (Greenspan 1968; Reisenegger 1993): an Ekman boundary layer, with $(\mathbf{v}_n)_\theta \sim R_* \Delta\Omega$, develops on a time-scale $\sim 2\pi/\Omega_*$, and grows radially to a thickness $d_{\rm E} \approx Re^{-1/2} R_*$ cm on a time scale $t_{\rm E} \approx Re^{1/2}(\Omega_*)^{-1}$, spinning up the interior fluid (Andersson 2003; Andersson et al. 2005). Only the normal fluid is Ekman pumped directly; the superfluid is spun up by the normal fluid through the mutual friction force (Adams et al. 1985; Reisenegger 1993).

In order to estimate the counterflow velocity, we compute $(\mathbf{v}_{ns})_z$ empirically from numerical experiments and scale up the results to neutron star parameters. For example, in a typical set of runs with $\delta = 0.3$ (HV mutual friction), 0.4 (HV mutual friction), and 0.5 (GM mutual friction), and $Re = 3 \times 10^4$, we find $(\mathbf{v}_n)_z \approx 3.15(\mathbf{v}_s)_z$, $(\mathbf{v}_n)_z \approx 0.78(\mathbf{v}_s)_z$ and $(\mathbf{v}_n)_z \approx 0.06(\mathbf{v}_s)_z$, respectively, on average over the grid, implying $(\mathbf{v}_{ns})_z \sim (\mathbf{v}_n)_z \sim R_* \Delta\Omega_{\rm em}$, as a general rule and hence

$$(\mathbf{v}_{ns})_z = 3.16 \left(\frac{\dot{\Omega}_*}{10^{-13}\,{\rm rad\,s^{-2}}} \right) \left(\frac{t}{1\,{\rm yr}} \right) \,{\rm cm\,s^{-1}} \tag{8}$$

In a typical neutron star, the Reynolds number in the outer core (temperature T) is very large, viz.

$$\begin{aligned} Re &= 1.67\times 10^{8}\left(\frac{\rho_n}{10^{15}\,\mathrm{g\,cm^{-3}}}\right)^{-1}\left(\frac{T}{10^{8}\,\mathrm{K}}\right)^{2} \\ &\quad\times\left(\frac{\Omega_*}{10^{2}\,\mathrm{rad\,s^{-1}}}\right), \end{aligned} \tag{9}$$

given the standard viscosity ν_n resulting from electron-electron scattering (Flowers & Itoh 1979; Cutler & Lindblom 1987; Andersson et al. 2005). Consequently, the flow is likely to be turbulent (Alpar 1978). Our numerical simulations cannot access this regime due to computational limitations (Peralta et al. 2005). However, we know from laboratory experiments with He II that the flow at $Re \gtrsim 10^5$ closely resembles classical, Navier-Stokes turbulence: the superfluid and normal fluid vorticity evolve in concert, because the vortex lines are locked to the normal fluid eddies by mutual friction (Ashton & Northby 1975; Barenghi et al. 1997). [5]

For the DGI to be triggered in superfluid turbulence, two conditions must be met. First, there must be vortex segments directed parallel to the counterflow, i.e. $\boldsymbol{\omega}_s \cdot \mathbf{v}_{ns} \neq 0$ (Barenghi et al. 1997). As the superfluid is locked to the normal fluid, and the vortex density is high, the condition $\boldsymbol{\omega}_s \cdot \mathbf{v}_{ns} \neq 0$ is satisfied locally throughout the core. Second, a vortex segment is locally unstable to (helical) Kelvin wave perturbations of wavelength λ if the local counterflow exceeds the critical speed $v_{\mathrm{DG}}^{\mathrm{turb}} = \kappa \ln(\lambda/2\pi a_0)/(2\lambda)$ (Barenghi et al. 1997). Large wavelengths (of the order of the size of the star) are most unstable, with $\lambda \sim 10^6$ cm giving $v_{\mathrm{DG}}^{\mathrm{turb}} = 3\times 10^{-2}$ cm s^{-1}. In this regard, the DGI threshold (6) is conservative ($v_{\mathrm{DG}}^{\mathrm{turb}} < v_{\mathrm{DG}}$). [6]

3.2. Glitch: decay of the tangle

During a glitch event, the differential rotation instantaneously shuts off and the tangle starts to decay. In a self-sustaining vortex tangle, Kelvin waves grow until their amplitude

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[5] In He II experiments, it is possible to reach a vortex density $\sim 10^7$ cm^{-2} (Skrbek et al. 2000), comparable to that in a neutron star, viz. $10^5(\Omega_*/10^2\,\mathrm{rad\,s^{-1}})$ cm^{-2} (Baym et al. 1992; Lyne & Graham-Smith 2006).

[6] A vacillating flow (Barenghi et al. 2004) can also trigger vortex oscillations by inducing an oscillating counterflow. A perturbation with wavelength λ grows if the vacillation frequency ω_0 is mantained over a time-scale $t \leq \lambda v_{ns}/2\pi\nu_s\omega_0$ (Barenghi et al. 2004).

approaches the average vortex separation, whereupon the vortex lines continuously reconnect to form loops (Jou & Mongiovì 2004). Reconnection stops once the counterflow ceases, as happens immediately after a glitch, when the outer core and inner crust come into corotation ($\Omega_1 = \Omega_2$). The decay time-scale $\tau_{\rm d}$ equals, to a good approximation, the reconnection time-scale just before the counterflow ceases.

To estimate $\tau_{\rm d}$, we approximate the vortex loops by rings of radius R, whose characteristic lifetime is given by $\tau_{\rm d} = R^2(2\nu_s\alpha)^{-1}$, with $\alpha = B\rho_n/2\rho$ (Barenghi et al. 1983; Tsubota et al. 2004). The radius of a ring can be estimated from the vortex line density L (length per unit volume), with $R = 0.5L^{-1/2}$ in a steady-state tangle. The evolution of L is given by Vinen's equation (Vinen 1957), generalized by Jou & Mongiovì (2004) to include rotation:

$$\frac{1}{\Omega_2}\frac{dX}{dt} = -\alpha_3 X^2 + \left[\frac{\alpha_1 v_{ns}}{(\kappa\Omega_2)^{1/2}} + \beta_2\right] X^{3/2} - \left[\beta_1 + \frac{\beta_4 v_{ns}}{(\kappa\Omega_2)^{1/2}}\right] X, \tag{10}$$

with $X = \kappa L/\Omega_2$, where α_i, β_i are dimensionless friction coefficients, whose values depend on the temperature (Mongiovì & Jou 2005). By fitting to experimental data, one finds $\alpha_3 = 20.0\alpha_1$, $\beta_1 = 35.6\alpha_1$, $\beta_2 = 53.6\alpha_1$, and $\beta_4 = 1.43\alpha_1$ (Swanson et al. 1983; Jou & Mongiovì 2004). From experiments, we also have $\alpha = \chi_1\alpha_1$ and $\chi_1 \sim 1$ (Vinen 1957), so we take $\alpha = \alpha_1$ below. Setting $dX/dt = 0$ in (10) and solving for L in the steady state, we find that the tangle decays on a time-scale

$$\tau_{\rm d} = \frac{2.5 \times 10^3 \kappa}{\alpha v_{ns}^2 \ln(b_0/a_0)} \tag{11}$$

$$\approx 7.6 \times 10^5 \left(\frac{\alpha}{10^{-7}}\right)^{-1} \times \left(\frac{\dot{\Omega}_*}{10^{-13}\,{\rm rad\,s^{-2}}}\right)^{-2} \left(\frac{t}{1\,{\rm yr}}\right)^{-2} {\rm s}, \tag{12}$$

where v_{ns} is the counterflow speed immediately before the differential rotation shuts off, given by (8).

Estimates for the friction parameter α are uncertain (Sonin 1987), but calculations based on electron scattering by proton vortex clusters and magnetized vortex cores give $B \approx 10^{-4}$ (Sauls et al. 1982; Sedrakian & Cordes 1998). [7] This implies $\alpha \lesssim 10^{-7}$ for $\rho/\rho_n = 10^{-2}$ in

[7] An absolute lower limit is $B = 10^{-18}$, calculated from electron scattering off vortex clusters in the high-

the outer core (Sedrakian & Sedrakian 1995). Therefore $\tau_{\rm d}$ is greater than the glitch trigger time-scale, as assumed a priori. Note that the B and B' parameters were also calculated by Mendell (1991) and Andersson et al. (2006), who found $B' = B^2$ and

$$\begin{aligned} B &\approx 4 \times 10^{-4} \left(\frac{m_p - m_p^*}{m_p} \right)^2 \left(\frac{m_p}{m_p^*} \right)^{1/2} \left(\frac{x_p}{0.05} \right)^{7/6} \\ & \times \left(\frac{\rho}{10^{14}\,\mathrm{g cm^{-3}}} \right)^{1/6} \end{aligned} \tag{13}$$

where $x_p = \rho/\rho_p$ is the proton fraction, and m_p and m_p^* are the bare and effective masses of the proton respectively (Andersson et al. 2006). Equation (13) includes modifications from entrainment (Mendell 1991; Andersson et al. 2006). In the outer core, with $\rho = 2.8 \times 10^{14}$ g cm^{-3} and $x_p = 0.038$ (Mendell 1991), we obtain $B = 2.7 \times 10^{-4}$, approximately the value quoted by Sauls et al. (1982) and Sedrakian & Cordes (1998).

In laboratory and numerical experiments, it is observed that the vortex tangle is polarized; the average vorticity projected along the rotation axis is not zero (Finne et al. 2003; Tsubota et al. 2004; Tsubota & Kasamatsu 2005). This is because, under these experimental conditions, one has $\beta_1 \gtrsim \beta_4 v_{ns}/(\kappa\Omega_2)^{1/2}$ in equation (10), where β_1 is the polarization-inducing term. Naively, one might expect the tangle to be even more polarized in a rapidly rotating neutron star, but in fact the opposite is true: the tangle is less polarized, because we have $\beta_1 \ll \beta_4 v_{ns}/(\kappa\Omega_2)^{1/2}$ from Figure 2. A thorough study of this issue, including whether the large-Ω_2 limit considered by Jou & Mongiovì (2004) in deriving equation (10) is applicable for $v_{ns} \gg (\kappa\Omega_2)^{1/2}$, lies outside the scope of this paper.

3.3. After a glitch: reformation of the tangle

After the tangle decays to a rectilinear array, differential rotation due to electromagnetic spin down between the inner and outer shells begins to grow until $(\mathbf{v}_{ns})_z$ exceeds $v_{\rm DG}$. From equations (6) and (8), this occurs after a time

$$t_{\rm tan} = 0.49 \left(\frac{\Omega_*}{10^2\,\mathrm{rad\,s^{-1}}} \right)^{1/2} \left(\frac{\dot{\Omega}_*}{10^{-13}\,\mathrm{rad\,s^{-2}}} \right)^{-1} \mathrm{yr}, \tag{14}$$

density regime, where the relaxation time-scale equals the age of the pulsar [Sedrakian & Cordes (1998); A. Sedrakian 2006, private communication].

whereupon a vortex tangle develops again in the outer core, via the DGI. This triggers a switch from HV to GM mutual friction, weakening the coupling between the normal fluid and superfluid components.

The characteristic time-scale for the tangle to develop equals the growth rate of helical vortex perturbations (Kelvin waves) via the DGI. By linearizing Schwarz's equation (Schwarz 1985, 1988), in the vortex filament model, one finds that the fastest growth occurs at a wavenumber $k = v_{ns}/2\nu_s$, with (Tsubota et al. 2004)

$$\tau_{\rm g} = \frac{\kappa \ln(b_0/a_0)}{\pi \alpha v_{ns}^2} \qquad (15)$$

$$\approx 4.9 \times 10^4 \left(\frac{\alpha}{10^{-7}}\right)^{-1} \left(\frac{\Omega_*}{10^2\,{\rm rad\,s^{-1}}}\right)^{-1} {\rm s}, \qquad (16)$$

where (16) follows from (15) by substituting (6). Hence, for typical neutron star parameters, the tangle is reestablished over ~ 1 d after ~ 1 yr elapses following a glitch, less than the inter-glitch interval observed in most pulsars (Shemar & Lyne 1996; Lyne et al. 2000). Note that $\tau_{\rm g}$ does not equal $\tau_{\rm d}$; the time required for a tangle to grow from a rectilinear array is shorter than the time required for an existing tangle to decay back to a rectilinear array, provided that $t < 8.0 t_{\rm tan}$ (and longer otherwise). Note also that v_{ns} just before the glitch ($\sim R_* \Delta\Omega$), which appears in (11), is typically greater than v_{ns} at $t_{\rm tan}$ ($\sim v_{\rm DG}$), which appears in (15), provided that $t > t_{\rm tan}$.

Equation (16) gives the minimum time for a tangle to reform, assuming it does so simultaneously everywhere in the outer core of the star. In practice, v_{ns} does not exceed $v_{\rm DG}$ everywhere simultaneously. Regions where the DGI is activated are interspersed with regions where it is not, even at $r = R_1$ and R_2, so that the transition from HV to GM friction is more gradual than equation (16) suggests. This important issue, which is also relevant to the interpretation of the laboratory experiments performed by Tsakadze & Tsakadze (1980), is explored briefly in §5. However, a thorough numerical study lies outside the scope of this paper.

4. Rotational evolution of the crust

In this section, we calculate numerically the torque acting on the stellar crust during a transition from a state of turbulent vorticity (tangle) to laminar vorticity (rectilinear array), by solving the hydrodynamic HVBK equations in a differentially rotating spherical shell. Global simulations of this type are necessary to calculate the observable response of a neutron star, e.g. $\Omega(t)$ and $\dot\Omega(t)$, to the internal physics elaborated in §3.

4.1. Numerical method and parameters

We solve equations (1) and (2) using a pseudospectral collocation method to discretize the problem (Boyd 2001; Canuto et al. 1988) and a time-split algorithm to step forward in time (Canuto et al. 1988), as described by Peralta et al. (2005). The velocity fields are expanded in a restricted Fourier series in θ and ϕ and a Chebyshev series in r (Orszag 1974; Boyd 2001), with $(N_r, N_\theta, N_\phi) = (120, 250, 4)$ modes required to fully resolve the flow. The initial time-step $\Delta t = 10^{-4}$ is lowered to $\Delta t = 10^{-5}$ during the GM $\rightarrow$ HV transition to prevent spurious oscillations in the torque (Peralta et al. 2005). Instabilities arising from the sensitivity of spectral methods to the boundary conditions (Peralta et al. 2005), and oscillations due to the Gibbs phenomenon (Gottlieb et al. 1984; Canuto et al. 1988), are smoothed using a low-pass spectral filter (Don 1994), a common practice (Osher 1984; Mittal 1999).

We adopt parameters as close to those of a realistic neutron star as our computational resources permit. In the outer core, we take $\rho_s/\rho = 0.99$ and $\rho_n/\rho = 0.01$ (Baym et al. 1992), and hence $B = 1.5$, $B' = 0.9$, and $A' = 1.0 \times 10^{-4}$ at the corresponding scaled temperature T/T_c in He II (Barenghi et al. 1983; Donnelly & Barenghi 1998). These friction coefficients are $\sim 10^4$ times greater than those used in the analytic estimates in Section 3. We adopt the higher values deliberately. Otherwise, the effects introduced by B and B' would take too long to build up; computational limitations prevent us from simulating more than ~ 10 rotation periods. (For the same reason, our $\Delta\Omega/\Omega$ is unrealistically large.) Realistic Reynolds numbers, estimated from equation (9), are too high to be simulated directly, so we restrict ourselves to the range $10^3 \lesssim Re \lesssim 10^5$; the most stable evolution occurs for $Re \approx 3 \times 10^4$, corresponding to an Ekman number $(Re)^{-1} \approx 3 \times 10^{-5}$ that is two orders of magnitude smaller than in a typical neutron star (Abney & Epstein 1996). The tension force is dominated by mutual friction, except in relatively old ($> 10^4$ yr) neutron stars (Greenstein 1970), with $\nu_s \sim 10^{-18}$ in our dimensionless units. For $10 \leq A \leq 10^4$, we find empirically that the torque depends weakly on the GM force, so we use $10 \leq A \leq 10^2$ to obtain identical results at lower computational cost.

4.2. Turbulent-laminar transition

We consider a turbulent initial state with $\Omega_1 > \Omega_2$ and $(\mathbf{v}_{ns})_z > v_{\mathrm{DG}}$ everywhere, such that the normal and superfluid components are coupled by GM mutual friction. The transition from a vortex tangle to a rectilinear vortex array is simulated by changing the mutual friction suddenly, from GM to HV, and simultaneously spinning up the outer shell, such that $\Omega_2 = \Omega_1$. This occurs at $t = 20$ in all the figures to follow. The angular shear before the transition lies in the range $0.1 \leq \Delta\Omega \leq 0.3$, for gaps in the range $0.2 \leq \delta \leq 0.5$. Note that we are forced to choose $\Delta\Omega$ unrealistically large, compared to typical neutron star values, in order to allow the inner and outer surfaces of the shell to "lap" each other within a reasonable run time.[8] Various combinations of $\Delta\Omega$ and δ affect the viscous torque on the inner and outer surfaces during steady differential rotation in different ways, as discussed below.

Streamline snapshots for a run with $\delta = 0.5$ and $\Delta\Omega = 0.2$ are presented in Figures 3a–3f, with the turbulent-laminar transition GM $\rightarrow$ HV, $\Omega_2 \rightarrow \Omega_1$ occurring at $t = 20$. The HV mutual friction, with $|\mathbf{F}_{\rm HV}|/|\mathbf{F}_{\rm GM}| \sim 10^5$, couples the normal and superfluid components strongly. [9] Both components exhibit a quasi-periodic motion, with a secondary circulation cell filling most of the shell, and two or three smaller vortices emerging intermittently. An oscillatory polar Ekman cell also exists initially, disappearing at $t = 140$.

We can compare this behaviour to the flow without a turbulent-laminar transition. Figures 4a–4f display a sequence of streamline snapshots for $\delta = 0.5$ and $\Delta\Omega = 0.2$, with GM mutual friction at all times. At $t = 18$, three polar cells are observed in both components, together with a secondary cell at mid-latitudes. The latter cell elongates after $t = 20$ and is present only in the superfluid component at later times. At the equator, the flow switches between one and three cells, but this behavior persists after $t = 100$ only in the superfluid component, while the normal fluid decouples increasingly.

The axial counterflow is significant at all times in Figures 4 and 5. Scaling to neutron star parameters, we find $v_{ns}/v_{\rm DG} \sim 10^3(\Delta\Omega/10^{-4}\,{\rm rad\,s^{-1}})(\Omega_*/10^2\,{\rm rad\,s^{-1}})$ near the equator, and $v_{ns}/v_{\rm DG} \sim 10^6(\Delta\Omega/10^{-4}\,{\rm rad\,s^{-1}})(\Omega_*/10^2\,{\rm rad\,s^{-1}})$ near the poles, at $t = 200$.

Figures 5a–5d display $\Delta\Omega/\Omega$ as a function of time after the turbulent-laminar transition for $\delta = 0.2$, 0.3, 0.4, and 0.5, respectively. [10] The evolution is qualitatively similar in all the cases considered. There is an initial transient, in which the flow adjusts to the initial spin up on a time-scale $t \sim 6$, just like in the classical (viscous) spin-up problem (Greenspan 1968). The fractional torque deviation, $\Delta\dot{\Omega}/\dot{\Omega}$, is plotted as a function of time in Figures 6a–6d. An initial jump, coinciding with the sudden spin up, is observed in all the panels,

[8]Flow structures associated with differential rotation develop on the time-scale $(\Delta\Omega)^{-1}$.

[9]The empirical result $|\mathbf{F}_{\rm HV}|/|\mathbf{F}_{\rm GM}| \sim 10^5$ is supported by order-of-magnitude estimates (Peralta et al. 2005) but should be treated with caution, because the microphysics of the GM friction force (and hence the value of A') has not yet been worked out rigorously in a neutron superfluid.

[10]Note that $\Delta\Omega/\Omega_*$ is not calculated self-consistently: it is obtained by numerically integrating the viscous torque on the outer sphere, after solving (1) and (2) subject to the boundary condition $\Omega_2(t) =$ constant, i.e. we do not include the (slight) evolution of $\Omega_2(t)$ when computing the fluid flow.

followed by a rapid exponential decay and persistent, small-amplitude oscillations. We find an e^{-1} relaxation time of $\tau \sim 2$, 1.6, and 1.6 for $\Delta\Omega = 0.1$, 0.2, and 0.3 respectively, with τ essentially independent of δ. In contrast, in the inverse experiment, where the mutual friction changes from HV to GM, Peralta et al. (2005) observed that $\Delta\dot{\Omega}$ is almost constant after the post-glitch transient. The period of the oscillations does not change with t for $t \gtrsim 100$.

4.3. Effect of the superfluid

The effect of the superfluid on the evolution can be analyzed with the help of Figure 7a, which plots the torque on the outer sphere as a function of time (solid curve) in the absence of spin up ($\delta = 0.5$, $Re = 3 \times 10^4$, $\Delta\Omega = 0.1$, GM friction). For comparison, the dashed curve plots the torque for a classical viscous fluid with the same parameters mentioned above. The evolution is qualitatively similar in both cases, displaying oscillations with the same periodicity and similar amplitude, but the curves diverge for $t \gtrsim 20$ as the viscous fluid evolves more rapidly to a stationary state, driven by classical Ekman pumping. Classical theory predicts a time-scale $t_{\rm E} \approx 173$ in the limit where the Rossby number $\Delta\Omega/\Omega$ vanishes; an exponential fit to the dashed curve gives $t_{\rm E} \approx 243$, which agrees well given that we have $\Delta\Omega/\Omega = 0.1$ in our numerical experiments. In a superfluid, on the other hand, the torque decays faster initially, with time constant ~ 50, then starts to increase again for $t \sim 240$, as part of a long-period oscillation whose ultimate fate is unknown as it occurs over a time-scale beyond our computational limit.

Figure 7b shows the evolution of the fractional change in angular velocity after spin up ($\Omega_2 = 0.9 \to 1.0$ at $t = 20$) in three scenarios: instantaneous change in mutual friction, GM $\to$ HV (solid curve); constant GM friction (dashed curve); and a classical viscous fluid (dashed dotted curve). The classical viscous fluid and constant GM friction evolve similarly. The torque decays exponentially on a time-scale ~ 2.15, then decays almost linearly with t. By contrast, in the GM $\to$ HV transition, the torque decays exponentially, on a time-scale ~ 2.20, then oscillates with peak-to-peak amplitude ≈ 0.4 (in units of $\rho R_2^5 \Omega_1^2$) and period ~ 6. In other words, the oscillations are sustained by the HV friction, which is much weaker than its GM counterpart. Note that the oscillations do not correspond to Tkachenko waves, since the HVBK equations assume that the free energy of the vortex array depends only on the vortex line density, not on vortex lattice deformations (Chandler & Baym 1983, 1986; Donnelly 1991).

4.4. Streamlines

The meridional streamlines for the transitions described in the previous paragraph evolve as in Figure 8. Figures 8a–8b show the streamlines for a classic viscous fluid before ($t = 20$) and after ($t = 22$) the glitch. There is a rapid redistribution of vorticity near the outer sphere, with two circulation cells replaced by one elongated meridional shell, while the flow near the inner sphere hardly changes, with one circulation cell near the equator and one near the pole. Figures 8c–8d show the meridional streamlines for the normal component of the superfluid. The flow resembles a classical viscous fluid (cf. Figure 8a), although it does become more complex after the spin up. As shown in Figure 8d, the large cell near the outer sphere is replaced by one primary circulation cell and three secondary cells (including one near the poles). Note that the greatest contributions to the torque come from the mid-latitude regions where the structure of eddies and counter-eddies is richer, as in Figure 8d. Finally, in Figures 8e–8f, we see the meridional streamlines for the inviscid component of the superfluid. The pattern before the jump (Figure 8e) differs from the normal fluid; the GM friction creates additional eddies near the outer sphere and closer to the poles. However, after the transition to HV friction, the inviscid component comes to resemble the normal component, as can be seen by comparing Figure 8f with Figure 8d. This occurs because the two components are coupled more strongly by HV friction ($|\mathbf{F}_{\rm HV}|/|\mathbf{F}_{\rm GM}| \sim 10^5$), so that the superfluid is dragged along by the normal fluid.

4.5. Stratification

Gravitational stratification can strongly suppress Ekman pumping, as the Ekman layer is squashed close to the outer sphere (Clark et al. 1971; Abney & Epstein 1996). Nevertheless, the Ekman layer, however thin, always exists, in order to satisfy the boundary conditions at $r = R_2$. It contains a meridional counterflow given by equation (8). Consequently, the transition from turbulent to laminar vorticity (and vice versa), which relies on such a meridional counterflow, still occurs in a stratified star, and its dramatic effect on the torque is the same. The Ekman layer, no matter how thin, acts like a film of "oil" between two sliding surfaces (here, the outer core and inner crust), whose coupling strength ("stickiness") changes abruptly when the meridional speed of the "oil" exceeds a threshold. In other words, equations (8), (12), (14), and (16), and the conclusions that follow from them, are unaltered by stratification; even though the volume of the outer core occupied by the vortex tangle (which does not affect the torque on the crust directly) does change.

The Ekman layer thickness decreases as $e^{-\kappa_Y} d_{\rm E}$, where $d_{\rm E}$ is the thickness without stratification and κ_Y is the compressibility of the fluid. The Ekman time-scale increases as

$(e^{\kappa_Y} - 1)\kappa_Y^{-1} t_{\rm E}$, where $t_{\rm E}$ is the time-scale without stratification. The fluid is restricted to move on concentric spherical shells (Levin & D'Angelo 2004). Importantly, $(e^{\kappa_Y} - 1)\kappa_Y^{-1} t_{\rm E}$ is the time for the Ekman layer to extend throughout the outer core, not the time required to establish the meridional flow at $r \approx R_2$ ($\sim \Omega_1^{-1}$), which controls the onset of the DGI.

It is conceivable that the DGI is excited in shells at certain radii where v_{ns} peaks, so that we end up with a sequence of shells containing alternating laminar and turbulent superfluid vorticity. In this scenario, the detailed study of which lies outside the scope of this paper, the stability of the vorticity configuration is restricted by the Richardson criterion, which states that the configuration is Kelvin-Helmholtz unstable for $N^2 |\partial v_\phi / \partial r|^{-2} < 1/4$ (Mastrano & Melatos 2005). In a neutron star, the Brunt-Väisäla frequency is $N \approx 5 \times 10^2\,{\rm s}^{-1}$ (Reisenegger & Goldreich 1992).

The effects of stratification are not considered in detail in this paper because they cannot be studied properly with our numerical method; the pressure projection step only works with divergence-free velocity fields (Bagchi & Balachandar 2002; Peralta et al. 2005). However, a crude approach to get a feel for the effects is to numerically suppress v_r using a low-pass exponential filter (Don 1994), viz. $v_r \to \exp[-(k/N_r)^\gamma \ln \epsilon] v_r$, with $0 \leq |k| \leq N_r$, where $\epsilon = 2.2 \times 10^{-16}$ is the machine zero, and γ is the order of the filter. If we ramp up γ with r as $\gamma = r(n_2 - n_1)/(R_2 - R_1) + n_1 - R_1(n_2 - n_1)/(R_2 - R_1)$, with $n_1 = 2$ and $n_2 = 12$, then v_r is weakly suppressed near R_2 ($\gamma = 12$) and strongly suppressed near R_1 ($\gamma = 2$), so that the flow is confined approximately to concentric shells. Preliminary results, for a viscous Navier-Stokes fluid, show that the filtering reduces the meridional circulation, flattening the streamlines radially.

5. Summary and discussion

In this paper, we investigate how transitions between turbulent and laminar states of superfluid vorticity alter the standard theoretical picture of pulsar rotational irregularities like glitches and timing noise. (i) Most of the time, except in the immediate aftermath of a glitch, differential rotation in the outer core drives a nonzero, poloidal counterflow which continuously excites the DGI. A vortex tangle is thereby maintained in the outer core. The mutual friction in this regime, which is of GM form, couples the normal and superfluid components loosely. (ii) Immediately after a glitch, the differential rotation ceases, as does the poloidal counterflow. The vortex tangle decays over the mean life-time of its constituent vortex rings, $\tau_{\rm d} = 7.6 \times 10^5 (\alpha/10^{-7})^{-1} (\dot{\Omega}_*/10^{-13}\,{\rm rad\,s^{-2}})^{-2} (t/1\,{\rm yr})^{-2}$ s. A rectilinear vortex array develops, and the mutual friction switches to HV form, coupling the normal and superfluid components much more strongly. (iii) After $t_{\rm tan} = 0.49\,(\Omega_*/10^2\,{\rm rad\,s^{-1}})^{1/2} (\dot{\Omega}_*/10^{-13}\,{\rm rad\,s^{-2}})^{-1}$

yr, electromagnetic spin down builds up the differential rotation sufficiently to drive a poloidal counterflow that exceeds the DGI threshold. A vortex tangle forms again in a time $\tau_{\rm g} = 4.9 \times 10^4 (\alpha/10^{-7})^{-1} (\dot{\Omega}_*/10^{-13}\,{\rm rad\,s^{-2}})^{-1}$ s, and the mutual friction reverts to GM form. Note that vortex pinning provides the boundary conditions for the superfluid SCF but does not occur within the outer core itself (Donati & Pizzochero 2003). *Therefore our new phenomenological picture is not a complete model for glitches.* It merely clarifies the vorticity state of the outer core before and after a glitch as an input into future models than incorporate the full glitch dynamics, including trigger mechanisms related to pinning in the inner crust.

We draw together the strands of the model in Figure 9, which displays the evolution of the torque and the regions where the DGI is active during the following numerical experiment: we fix $\Delta\Omega = 0.1$ until $t = 20$, accelerate the outer sphere instantaneously to corotation at $t = 20$, then decelerate the outer sphere according to $\Omega_2(t) = 1 - 0.001(t - 20)$ for $t > 20$. This mimics the situation in a real pulsar, where we have $t_{\rm E} \ll t_{\rm tan}$, i.e. Ekman pumping brings the fluid into corotation *before* the DGI gradually starts being reexcited throughout the outer core. To make the experiment as realistic as possible, we do not assume that the mutual friction takes the same form everywhere in the outer core, but rather choose GM or HV friction at each point according to whether $(\mathbf{v}_{ns})_z$ is greater or less than $v_{\rm DG}$ locally. In this comparison, we approximate $\mathbf{v}_{ns}$ by $\mathbf{v}_n$, as in equation (8), because $\mathbf{v}_s$ can become very complicated (e.g. Figure 1), creating numerical difficulties. In order to satisfy $t_{\rm E} \ll t_{\rm tan}$ while keeping $\Delta\Omega$ large enough so that the spheres "lap" each other several times, we are forced by computational exigencies to adopt a relatively low Reynolds number $Re = 100$, shortening the Ekman time ($t_{\rm E} \sim 10\Omega_1^{-1} \ll t_{\rm tan}$), and to artificially boost $v_{\rm DG}$, so that $|(\mathbf{v}_n)_z/v_{\rm DG}|$ does not exceed ~ 3 throughout the computational domain.

The results of the above numerical experiment are presented in Figure 9. Contours of $|(\mathbf{v}_n)_z/v_{\rm DG}|$, before and after the spin up at $t = 20$, are plotted in Figures 9a–9e. Shaded regions indicate where the DGI is active, i.e. $|(\mathbf{v}_n)_z/v_{\rm DG}| > 1$. Just before the glitch (Figure 9a), 32 % of the superfluid is in a turbulent state, with the DGI active close to the inner sphere and at intermediate latitudes where meridional circulation is significant. After the differential rotation shuts off at $t = 20$, the DGI initially spreads through the shell as transient axial flows increase, occupying 39 % of the volume at $t = 22$. However, the flow quickly settles down to a state of near-corotation during the time interval $24 \lesssim t \lesssim 50$, HV friction dominates, and the torque decays exponentially, with time constant ~ 10 (followed by a linear decay). At $t = 50$, the DGI slowly begins to assert itself again, starting from the inner sphere. As for a classical viscous fluid, when the outer sphere spins down, it pumps fluid radially inward and along the axis of rotation (Vanyo 1993), so the axial flow speed is greatest near the inner sphere. By $t = 120$, when $\Delta\Omega = 0.1$, the vorticity state is similar to that at $t = 20$, before spin up.

One might wonder if, in a realistic neutron star, the superfluid ever exits the turbulent state and becomes laminar. For the simulations in this paper, which have $Re \leq 3 \times 10^4$, the answer is yes. Figure 9 shows that, when the glitch occurs and the spheres come into corotation, $|(\mathbf{v}_n)_z|$ falls below the DGI threshold after a time $t \sim t_{\mathrm{E}}$. The vortex tangle is then guaranteed to decay on a time-scale given by (11) and (12), as observed in terrestrial experiments. However, for more realistic neutron star Reynolds numbers ($Re \geq 10^8$), which are too challenging to simulate at present, the turbulent eddies in the normal fluid decay more slowly when the spheres come into corotation, so that $|(\mathbf{v}_n)_z|$ remains above the DGI threshold for longer. If this happens, the vortex tangle may persist until the next glitch occurs, so that the superfluid never exits the turbulent state.

What are the implications of Figure 9 and the results in §3 and §4 for observations of glitches? Before considering this question, we emphasize again that the results in this paper do not constitute a theory of glitches, because important questions regarding the glitch trigger remain unresolved. Nevertheless, some general remarks can be made. First of all, it is clear that transitions between flow (and vorticity) states in superfluid SCF are caused by changes in Re, and that such transitions become more frequent and complicated as Re increases (Yavorskaya et al. 1977; Junk & Egbers 2000). This is compatible with the observation that adolescent pulsars ($\sim 10^4$ yr old, like Vela) glitch most actively (Lyne et al. 2000). In younger pulsars (age $\lesssim 10^4$ yr), T and hence ν_n are relatively high, so Re is low. In older pulsars (age $\gtrsim 10^4$ yr), Ω and hence Re are low following electromagnetic spin down [although this trend is not straightforward and can be masked by localized heating from differential rotation between the superfluid and the crust (Greenstein 1975; Larson & Link 1999) or crust cracking (Link et al. 1998; Franco et al. 2000)]. A systematic statistical study of glitch activity versus Re will be published elsewhere (Melatos et al. 2006), but preliminary estimates of T and hence Re from cooling curves (Tsuruta 1974, 1998; Page et al. 2004) including superfluidity (Flowers & Itoh 1976, 1979; Andersson et al. 2005) give Reynolds numbers in the range $10^8 \lesssim Re \lesssim 10^{12}$ for glitching pulsars. Two of the most active glitchers, the Crab and Vela, have $Re \sim 10^9$ and $Re \sim 10^{10}$ respectively. One expects that, at such high Re, the fluid is turbulent, with the kinetic energy concentrated at large scales (Yavorskaya et al. 1978, 1986), as for a classical viscous fluid (Smith et al. 1993; Barenghi et al. 1997). This suggests that superfluid turbulence in pulsar interiors is an important factor in glitch dynamics.

If it is true that the vorticity in the outer core exists in a turbulent state before a glitch, as postulated in our model, then $t_{\tan}$ represents a lower bound on the time between glitches. In testing whether this bound is respected by the glitching pulsars currently known, we are hampered by the fact that most of these objects have only glitched once. Nevertheless, for all the 28 pulsars that have glitched repeteadly, we find that the minimum inter-glitch time interval $t_{\rm min}$ is greater than $t_{\rm tan}$, as the theory predicts (Melatos et al. 2006). The object PSR $2116 + 1414$ approaches the bound most closely, with $t_{\rm min} = 3.9$ yr and $t_{\rm tan} = 2.0$ yr. This is encouraging, because the 28 objects cover five decades in $t_{\rm tan}$ and three decades in $t_{\rm min}$, and the theoretical expression (14) for $t_{\rm tan}$ contains zero free parameters. Note that $t_{\rm tan}$ is proportional to the characteristic age ($= \Omega_*/2\dot{\Omega}_*$) divided by $\Omega_*^{1/2}$. Note also that the activity parameter defined by McKenna & Lyne (1990) involves glitch amplitudes (which are highly variable) as well as mean recurrence times, so we do not predict a correlation between the activity parameter and $t_{\rm tan}$.

It is harder to test the theoretical decay time-scale of the vortex tangle, as predicted by (12), because it remains unclear what observable features are engendered by the decay process. The observed exponential post-glitch relaxation is of viscous origin and occurs on a time-scale much larger than $\tau_{\rm d}$. On the other hand, the decay of the tangle is accompanied by a large increase in mutual friction (GM $\rightarrow$ HV), which may be connected with the rapid jump in Ω during a glitch. The jump in Ω has never been resolved in time, in pulsars which are nearly constantly monitored, consistent with the predictions of (12) for the Crab ($\tau_{\rm d} = 3 \times 10^{-4}$ s) and Vela ($\tau_{\rm d} = 0.2$ s). However, equation (12) predicts that it may be possible to resolve the Ω jump in older pulsars, provided that the time between glitches does not increase faster than $\dot{\Omega}_*$. In making these estimates we assume the canonical value $\alpha = 10^{-7}$ for every object, in the absence of a microscopic theory, yet this is clearly an oversimplification because α is sensitively temperature dependent.

Oscillations in $\dot{\Omega}_*$ were observed before (period $\sim$ 10 d) and after (period $\sim$ 25 d) the Vela Christmas glitch, with $\Delta\dot{\Omega}/\dot{\Omega}_* \approx 0.17$ (McCulloch et al. 1990). In our numerical simulations, persistent torque oscillations are always observed when the outer core rotates differentially, as occurs before a glitch. They are also observed after a switch from GM to HV friction, as occurs after a glitch. By comparing the dashed and solid curves in Figure 7b, we see that the oscillations are sustained by HV mutual friction. The oscillation period in our simulations is much shorter than in pulsar data, because we are restricted to $Re \leq 3 \times 10^4$. An alternative explanation is that vortices in the inner crust oscillate relative to the normal fluid in the core (Sedrakian et al. 1995).

Several of the effects explored in this paper have been studied in terrestrial laboratories. Our results will motivate new experiments of this sort, cf. Alpar (1978) and Anderson et al. (1978). Although it is hard to access the neutron star regime $\rho_n \ll \rho_s$ in He II, where inter-atomic forces are appreciable, transitions between turbulent and laminar superfluid vorticiy have been observed in experiments with microspheres immersed in ^{4}He at mK temperatures

(Niemetz et al. 2002). Promising results on the relaxation of rotating He II were obtained by Tsakadze & Tsakadze (1980), but again these results are for $\rho_n \lesssim \rho_s$ and hollow spheres rather than a differentially rotating shell. We propose to extend these experiments in two directions: (i) by investigating low-Rossby-number ($\Delta\Omega/\Omega \ll 1$), high-Reynolds-number ($Re \gg 10^5$) SCF with He II at the temperature which minimizes ρ_n/ρ_s; and (ii) by repeating (i) with a nonideal dilute-gas Bose-Einstein condensate confined in a differentially rotating magneto-optical trap, in order to probe the stability of a vortex lattice to Kelvin wave excitations in the regime $\rho_n \ll \rho_s$ (Parker & Adams 2005). The presence of a vortex tangle in He II can be detected by standard second-sound absorption techniques (Hall & Vinen 1956a; Swanson et al. 1983), and the torque in experiment (i) can be monitored to look for oscillations when a change from HV to GM friction (or vice versa) is triggered by the DGI.

In classical Navier-Stokes fluids, injection of vorticity into a metastable laminar state can trigger turbulence, e.g. seed vortices injected into a cylindrical vessel containing ^{3}He-B (with $T \leq 0.6T_c$) generate a vortex tangle that eventually decays into a rectilinear vortex array (Finne et al. 2003). Unlike He II, the normal component in ^{3}He-B is laminar in these experiments and does not participate in the turbulent dynamics, due to its comparatively high viscosity [$\nu_n \sim 1$ cm^2 s$^{-1} \gg \nu_s$, cf. $\nu_n \sim \nu_s$ in He II; see Finne et al. (2004)]. Standard glitch theories assume that the normal component is tightly coupled to the crust by the external magnetic field (Alpar & Sauls 1988; Jahan Miri 1998). This suggest a second possible SCF experiment with ^{3}He-B (at $T \leq 0.6T_c$) in a hollow spherical container in which the normal fluid corotates with the container, and therefore does not participate in the DGI, while the superfluid is free to become turbulent. Such an experiment can probe what aspects of the turbulent-laminar transition are caused by the normal and superfluid components respectively. Transitions between a vortex tangle and a rectilinear array can be detected using non-invasive nuclear magnetic resonance techniques (Finne et al. 2003).

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Fig. 1.— Streamlines in superfluid spherical Couette flow with $\delta = 0.4$, $Re = 3 \times 10^4$, and $\Delta\Omega = 0.1$. (a) Normal fluid with GM mutual friction. (b) Superfluid with GM mutual friction. (c) Normal fluid with HV mutual friction. (d) Superfluid with HV mutual friction. The streamlines are calculated by integrating the in-plane components of the velocity fields in the plane $x = 0$ at $t = 18$.

Fig. 2.— Normalized counterflow velocity $(\mathbf{v}_{ns})_z/v_{\rm DG}$ in superfluid spherical Couette flow with $\delta = 0.4$, $Re = 3 \times 10^4$, and $\Delta\Omega = 0.1$ at $t = 46$. (a) HV mutual friction. (b) GM mutual friction.

Fig. 3.— Meridional streamlines of the normal fluid (left) and superfluid (right) components, for $\delta = 0.5$, after the outer sphere is accelerated instantaneously from $\Omega_2 = 0.8$ at $t < 20$ to $\Omega_2 = 1$ at $t \geq 20$, and the mutual friction is changed simultaneously from GM to HV. The snapshots correspond to (a) $t = 21$, (b) $t = 22$, (c) $t = 50$, (d) $t = 100$, (e) $t = 120$, and (f) $t = 140$. Time is expressed in units of Ω_1^{-1}.

Fig. 4.— Meridional streamlines of the normal fluid (left) and superfluid (right) components as a function of time for $\delta = 0.5$, $\Delta\Omega = 0.2$, and GM mutual friction. There is no spin-up event at $t = 20$, unlike in Figure 3. The snapshots correspond to (a) $t = 18$, (b) $t = 20$, (c) $t = 50$, (d) $t = 100$, (e) $t = 120$, and (f) $t = 140$. Time is expressed in units of Ω_1^{-1}.

Fig. 5.— Fractional change in angular velocity of the outer sphere, $\Delta\Omega/\Omega = [\Omega_2(t) - \Omega_2(20)]/\Omega_2(20)$, as a function of time, before and after a spin-up event at $t = 20$ where the mutual friction is changed instantaneously from GM to HV and Ω_2 jumps according to $\Omega_2 = 0.9 \rightarrow 1$ (solid curve), $\Omega_2 = 0.8 \rightarrow 1$ (dashed curve), and $\Omega_2 = 0.7 \rightarrow 1$ (dotted-dashed curve). Time is measured in units of Ω_1^{-1}. Dimensionless gap width: (a) $\delta = 0.2$, (b) $\delta = 0.3$, (c) $\delta = 0.4$, and (d) $\delta = 0.5$.

Fig. 6.— Fractional change in the angular acceleration of the outer sphere, $\Delta\dot{\Omega}/\dot{\Omega} = [\dot{\Omega}_2(t) - \dot{\Omega}_2(20)]/\dot{\Omega}_2(20)$, as a function of time, before and after a spin-up event at $t = 20$ where the mutual friction is changed instantaneously from GM to HV and Ω_2 jumps according to $\Omega_2 = 0.9 \rightarrow 1$ (solid curve), $\Omega_2 = 0.8 \rightarrow 1$ (dashed curve), and $\Omega_2 = 0.7 \rightarrow 1$ (dotted-dashed curve). Time is measured in units of Ω_1^{-1}. Dimensionless gap width: (a) $\delta = 0.2$, (b) $\delta = 0.3$, (c) $\delta = 0.4$, and (d) $\delta = 0.5$.

Fig. 7.— (a) Evolution of the z component of the torque on the outer sphere (multiplied by 10^4) as a function of time for a superfluid with GM mutual friction (solid curve) and a classical Navier-Stokes fluid (dashed curve), with $\delta = 0.5$, $Re = 3 \times 10^4$, and $\Delta\Omega = 0.1$. The torque is expressed in units of $\rho R_2^5 \Omega_1^2$ and the time in units of Ω_1^{-1}. (b) Fractional change in the angular acceleration $\Delta\dot{\Omega}/\dot{\Omega} = [\dot{\Omega}_2(t) - \dot{\Omega}_2(20)]/\dot{\Omega}_2(20)$ following the spin-up event $\Omega_2 = 0.9 \rightarrow 1.0$ at $t = 20$ with $\delta = 0.5$ and $Re = 3 \times 10^4$, in three cases: GM $\rightarrow$ HV transition (dashed curve), superfluid with GM mutual friction (solid curve), and classical Navier-Stokes fluid (dashed-dotted curve).

Fig. 8.— Meridional streamlines for superfluid spherical Couette flow with $\delta = 0.5$ and $Re = 3 \times 10^4$, obtained by integrating the in-plane velocity components in the plane $x = 0$. (a) Viscous fluid at $t = 20$, with $\Omega_2 = 0.9$ and $\Omega_1 = 1.0$, and (b) at $t = 22$, after a sudden spin up of the outer sphere $\Omega_2 = 0.9 \rightarrow 1.0$. (c) Viscous normal component at $t = 20$, with $\Omega_2 = 0.9$, $\Omega_1 = 1.0$, and GM friction, and (d) at $t = 22$, after a sudden spin-up of the outer sphere $\Omega_2 = 0.9 \rightarrow 1.0$, while simultaneously changing the friction from GM to HV. (e) Inviscid superfluid component at $t = 20$, with $\Omega_2 = 0.9$, $\Omega_1 = 1.0$, and GM friction, and (f) at $t = 22$, after a sudden spin up of the outer sphere $\Omega_2 = 0.9 \rightarrow 1.0$, while simultaneously changing the friction force from GM to HV. Time is measured in units of Ω_1^{-1}.

Fig. 9.— Turbulent-laminar vorticity transition during a glitch. Evolution of the torque on the outer sphere before and after the outer sphere is impulsively accelerated from $\Omega_2 = 0.9$ to $\Omega_2 = 1.0$ at $t = 20$, for $\delta = 0.5$. The angular velocity of the outer sphere is $\Omega_2 = 0.9$ at $0 \leq t < 20$, and $\Omega_2(t) = 1.0 + 0.001(t - 20)$ during the time interval $20 \leq t \leq 120$. The five meridional slices are contour plots of $|(\mathbf{v}_n)_z/v_{\mathrm{DG}}|$ at (a) $t = 20$, (b) $t = 22$, (c) $t = 50$, (d) $t = 70$, and (e) $t = 120$. Dark regions indicate where the DGI is active, i.e. $|(\mathbf{v}_n)_z/v_{\mathrm{DG}}| \geq 1.0$, white regions indicate where the DGI is not active, i.e. $|(\mathbf{v}_n)_z/v_{\mathrm{DG}}| < 1.0$, and the mutual friction is of HV form.